

## THE RELATION BETWEEN HENSTOCK INTEGRAL AND HENSTOCK DELTA INTEGRAL ON TIME SCALES

JAE MYUNG PARK\*, DEOK HO LEE\*\*, JU HAN YOON\*\*\*, YOUNG  
KUK KIM\*\*\*\*, AND JONG TAE LIM\*\*\*\*\*

ABSTRACT. In this paper, we define an extension  $f^* : [a, b] \rightarrow \mathbb{R}$  of a function  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  for a time scale  $\mathbb{T}$  and show that  $f$  is Henstock delta integrable on  $[a, b]_{\mathbb{T}}$  if and only if  $f^*$  is Henstock integrable on  $[a, b]$ .

### 1. Introduction and preliminaries

The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [6].

In this paper, we investigate the relation between the Henstock integral and Henstock delta integral on time scales.

First, we introduce some concepts related to the notion of time scales. A time scale  $\mathbb{T}$  is any closed nonempty subset of  $\mathbb{R}$ , with the topology inherited from the standard topology on the real numbers  $\mathbb{R}$ . For each  $t \in \mathbb{T}$ , we define the forward jump operator  $\sigma(t)$  by

$$\sigma(t) = \inf\{z > t : z \in \mathbb{T}\}$$

and the backward jump operator  $\rho(t)$  by

$$\rho(t) = \sup\{z < t : z \in \mathbb{T}\}$$

where  $\inf \phi = \sup \mathbb{T}$  and  $\sup \phi = \inf \mathbb{T}$ .

If  $\sigma(t) > t$ , we say the  $t$  is right-scattered, while if  $\rho(t) < t$ , we say that  $t$  is left-scattered. If  $\sigma(t) = t$ , we say that  $t$  is right-dense, while if  $\rho(t) = t$ , we say that  $t$  is left-dense. The forward graininess function

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Correspondence should be addressed to Ju Han Yoon, yoonjh@chnu.ac.kr.

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$\mu(t)$  is defined by  $\mu(t) = \sigma(t) - t$ , and the backward graininess function  $\nu(t)$  is defined by  $\nu(t) = t - \rho(t)$ .

For  $a, b \in \mathbb{T}$ , we define the time scale interval in  $\mathbb{T}$  by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

## 2. The Henstock and Henstock delta integrals

DEFINITION 2.1. ([6])  $\delta = (\delta_L, \delta_R)$  is a  $\Delta$ -gauge on  $[a, b]_{\mathbb{T}}$  by  $\delta_L(t) > 0$  on  $(a, b]_{\mathbb{T}}$ ,  $\delta_R(t) > 0$  on  $[a, b)_{\mathbb{T}}$ ,  $\delta_L(a) \geq 0$ ,  $\delta_R(b) \geq 0$ , and  $\delta_R(t) \geq \mu(t)$  for each  $t \in [a, b]_{\mathbb{T}}$ .

DEFINITION 2.2. ([6]) A collection  $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^n$  of tagged intervals is a Henstock partition of  $[a, b]_{\mathbb{T}}$  if  $\bigcup_{i=1}^n [t_{i-1}, t_i]_{\mathbb{T}} = [a, b]_{\mathbb{T}}$ ,  $[t_{i-1}, t_i]_{\mathbb{T}} \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$  and  $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}}$  for each  $i = 1, 2, \dots, n$ .

For Henstock partition  $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^n$ , we write

$$S(f, \mathcal{P}) = \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}),$$

whenever  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ .

DEFINITION 2.3. ([6]). A function  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  is Henstock delta integrable (or  $H_{\Delta}$ -integrable) on  $[a, b]_{\mathbb{T}}$  if there exists a number  $A$  such that for each  $\epsilon > 0$  there exists a  $\Delta$ -gauge  $\delta$  on  $[a, b]_{\mathbb{T}}$  such that

$$\left| S(f, \mathcal{D}) - A \right| < \epsilon$$

for every  $\delta$ -fine Henstock partition  $\mathcal{D}$  of  $[a, b]_{\mathbb{T}}$ . The number  $A$  is called the  $H_{\Delta}$ -integral of  $f$  on  $[a, b]_{\mathbb{T}}$ , and we write  $A = (H_{\Delta}) \int_a^b f \Delta t$ .

Recall that  $f : [a, b] \rightarrow \mathbb{R}$  is Henstock integrable (or H-integrable) on  $[a, b]$  if there exists a number  $A$  such that for each  $\epsilon > 0$  there exists a gauge  $\delta : [a, b] \rightarrow \mathbb{R}^+$  on  $[a, b]$  such that

$$\left| S(f, \mathcal{P}) - A \right| < \epsilon$$

for every  $\delta$ -fine Henstock partition  $\mathcal{P}$  of  $[a, b]$ .

The proof of the following theorem can be found in [5].

THEOREM 2.4. [5] *A function  $f : [a, b] \rightarrow \mathbb{R}$  is H-integrable on  $[a, b]$  if and only if  $f$  is  $H_{\Delta}$ -integrable on  $[a, b]$ .*

Let  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  be a function on  $[a, b]_{\mathbb{T}}$ , and let  $\{(a_k, b_k)\}_{k=1}^{\infty}$  be the sequence of intervals contiguous to  $[a, b]_{\mathbb{T}}$  in  $[a, b]$ .

Define a function  $f^* : [a, b] \rightarrow \mathbb{R}$  on  $[a, b]$  by

$$f^*(t) = \begin{cases} f(a_k) & \text{if } t \in (a_k, b_k) \text{ for some } k \\ f(t) & \text{if } t \in [a, b]_{\mathbb{T}} \end{cases}$$

Then we have the following theorem.

**THEOREM 2.5.** [5] *If  $f^* : [a, b] \rightarrow \mathbb{R}$  is  $H$ -integrable on  $[a, b]$ , then  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  is  $H_{\Delta}$ -integrable on  $[a, b]_{\mathbb{T}}$  and  $(H_{\Delta}) \int_a^b f \Delta t = (H) \int_a^b f^*$ .*

Now we can prove the following theorem.

**THEOREM 2.6.** *If  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  is  $H_{\Delta}$ -integrable, then  $f^* : [a, b] \rightarrow \mathbb{R}$  is  $H$ -integrable and  $(H) \int_a^b f^* = (H_{\Delta}) \int_a^b f \Delta t$ .*

*Proof.* Let  $\epsilon > 0$ . Since  $f$  is  $H_{\Delta}$ -integrable, there exists a  $\Delta$ -gauge  $\delta = (\delta_L, \delta_R)$  on  $[a, b]_{\mathbb{T}}$  with  $\delta_R(t) = \sigma(t) - t$  if  $t$  is a right-scattered point of  $[a, b]_{\mathbb{T}}$  such that

$$\left| f(Q) - (H_{\Delta}) \int_a^b f \Delta t \right| < \epsilon$$

for each  $\delta$ -fine partition of  $[a, b]_{\mathbb{T}}$ .

Let  $\{(a_k, b_k)\}_{k=1}^{\infty}$  be the sequence of intervals contiguous to  $[a, b]_{\mathbb{T}}$  in  $[a, b]$ . Define a gauge  $\delta^* = (\delta_L^*, \delta_R^*)$  on  $[a, b]$  as follows;

For each left-dense point  $t$  of  $[a, b]_{\mathbb{T}}$ , choose  $s_1 \in [a, b]_{\mathbb{T}}$  such that  $t - \delta_L(t) \leq s_1 < t$  and let  $\delta_L^*(t) = t - s_1$ .

For each right-dense point  $t$  of  $[a, b]_{\mathbb{T}}$ , choose  $s_2 \in [a, b]_{\mathbb{T}}$  such that  $t < s_2 \leq t + \delta_R(t)$  and let  $\delta_R^*(t) = s_2 - t$ .

For each  $k \in \mathbb{N}$ , let  $\delta_R^*(a_k) = \frac{b_k - a_k}{3}$  and

$$\delta_L^*(b_k) = \min \left\{ \frac{\epsilon}{2^k(|f(a_k)| + |f(b_k)| + 1)}, \frac{b_k - a_k}{3}, \delta_L(b_k) \right\}.$$

For each  $t \in (a_k, b_k)$ , let  $\delta_L^*(t) = \frac{1}{2}(t - a_k)$  and  $\delta_R^*(t) = \frac{1}{2}(b_k - t)$ ,  $k = 1, 2, 3, \dots$ .

Assume that  $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$  is a  $\delta^*$ -fine partition of  $[a, b]$ . By separating the tagged interval  $(\xi_i, [t_{i-1}, t_i])$  into  $(\xi_i, [t_{i-1}, \xi_i])$  and  $(\xi_i, [\xi_i, t_i])$  if necessary, we may assume that  $\xi_i = t_{i-1}$  or  $\xi_i = t_i$  for each  $i = 1, 2, \dots, n$ .

Let  $K = \{k \in \mathbb{N} \mid (a_k, b_k) \cap \{t_1, t_2, \dots, t_{n-1}\} \neq \emptyset\}$  and let  $K$  be ordered as  $K = \{k_1, k_2, k_3, \dots, k_v\}$  such that  $a_{k_1} < a_{k_2} < \dots < a_{k_v}$ . For each  $j(1 \leq j \leq v)$ , denote

$$p_j = \min\{i \mid t_i \in (a_{k_j}, b_{k_j}), 1 \leq j \leq n\} \quad \text{and} \\ q_j = \max\{i \mid t_i \in (a_{k_j}, b_{k_j}), 1 \leq j \leq n\}.$$

Obviously, we have  $\xi_i \in [a, b]_{\mathbb{T}}$  if  $t_{i-1} \in [a, b]_{\mathbb{T}}$  and  $t_i \in [a, b]_{\mathbb{T}}$ .

Let  $Q_0 = \{(\xi_i, [t_{i-1}, t_i]) \mid t_{i-1} \in [a, b]_{\mathbb{T}} \text{ and } t_i \in [a, b]_{\mathbb{T}}\}$ . Then  $Q_0$  is a  $\delta$ -fine partial partition of  $[a, b]_{\mathbb{T}}$ . Note that  $\xi_{p_j} = t_{p_j-1} \in [a, b]_{\mathbb{T}}$  and  $\xi_{q_j+1} = t_{q_j+1} \in [a, b]_{\mathbb{T}}$  for each  $j (1 \leq j \leq v)$ .

Let  $\mathcal{P}_j = \{(\xi_i, [t_{i-1}, t_i]) \mid p_j \leq i \leq q_{j+1}\}$  for each  $j = 1, 2, \dots, v$ . Then  $f^*(\mathcal{P}) = f(Q_0) + \sum_{j=1}^v f^*(\mathcal{P}_j)$ .

First, assume that  $f^*(\mathcal{P}) \leq (H_{\Delta}) \int_a^b f \Delta t$ . Define  $Q_1, Q_2, \dots, Q_v$  as follows;

**Case 1.**  $t_{p_j-1} = a_{k_j}$  and  $t_{q_j+1} = b_{k_j}$ . Let  $Q_j = \{(\xi_{p_j}, [a_{k_j}, b_{k_j}])\}$ . Then  $Q_j$  is  $\delta$ -fine and

$$\begin{aligned} f^*(\mathcal{P}_j) &= f(a_{k_j})(t_{q_j} - a_{k_j}) + f(b_{k_j})(b_{k_j} - t_{q_j}) \\ &= f(a_{k_j})(b_{k_j} - a_{k_j}) + (f(b_{k_j}) - f(a_{k_j}))(b_{k_j} - t_{q_j}) \\ &\geq f(a_{k_j})(b_{k_j} - a_{k_j}) - (|f(b_{k_j})| + |f(a_{k_j})|)(b_{k_j} - t_{q_j}) \\ &> f(Q_j) - \frac{\epsilon}{2^{k_j}}. \end{aligned}$$

**Case 2.**  $t_{p_j-1} = a_{k_j}$  and  $b_{k_j} < t_{q_j+1}$ .

Let

$$Q_j = \begin{cases} \{(\xi_{q_j+1}, [a_{k_j}, t_{q_j+1}])\} & \text{if } f(a_{k_j}) \geq f(\xi_{q_j+1}) \\ \{(\xi_{p_j}, [a_{k_j}, b_{k_j}]), (\xi_{q_j+1}, [b_{k_j}, t_{q_j+1}])\} & \text{if } f(a_{k_j}) > f(\xi_{q_j+1}). \end{cases}$$

Then  $Q_j$  is  $\delta$ -fine and  $f^*(\mathcal{P}_j) \geq f(Q_j)$ .

**Case 3.**  $t_{p_j-1} < a_{k_j}$  and  $b_{k_j} = t_{q_j+1}$ .

In this case, we have

$$\begin{aligned} f^*(\mathcal{P}_j) &= f(\xi_{p_j})(t_{p_j} - t_{p_j-1}) + f(b_{k_j})(b_{k_j} - t_{q_j}) + f(a_{k_j})(t_{q_j} - t_{p_j}) \\ &\geq f(\xi_{p_j})(t_{p_j} - t_{p_j-1}) + f(a_{k_j})(b_{k_j} - t_{p_j}) - \frac{\epsilon}{2^{k_j}}. \end{aligned}$$

Let

$$Q_j = \begin{cases} \{(\xi_{p_j}), [t_{p_j-1}, b_{k_j}]\} & \text{if } f(\xi_{p_j}) \leq f(a_{k_j}) \\ \{(\xi_{p_j}), [t_{p_j-1}, a_{k_j}], (a_{k_j}, [a_{k_j}, b_{k_j}])\} & \text{if } f(\xi_{p_j}) > f(a_{k_j}). \end{cases}$$

Then  $f^*(\mathcal{P}_j) \geq f(Q_j) - \frac{\epsilon}{2^{k_j}}$ .

**Case 4.**  $t_{p_j-1} < a_{k_j}$  and  $b_{k_j} < t_{q_j+1}$ .

Let  $m = \min\{f(\xi_{p_j}), f(a_{k_j}), f(\xi_{q_j+1})\}$ . Define

$$Q_i = \begin{cases} \{(\xi_{p_j}, [t_{p_j-1}, b_{k_j}]), (\xi_{q_j+1}, [b_{k_j}, t_{q_j+1}])\} & \text{if } f(\xi_{p_j}) = m \\ \{(\xi_{p_j}, [t_{p_j-1}, a_{k_j}]), (\xi_{q_j+1}, [a_{k_j}, t_{q_j+1}])\} & \text{if } f(\xi_{p_j}) = m \text{ and } f(a_{k_j}) \geq f(\xi_{q_j+1}) \\ \{(\xi_{p_j}, [t_{p_j-1}, a_{k_j}]), (a_{k_j}, [a_{k_j}, b_{k_j}]), (\xi_{q_j+1}, [b_{k_j}, t_{q_j+1}])\} & \text{if } f(\xi_{p_j}) \neq m \text{ and } f(a_{k_j}) < f(\xi_{q_j+1}) \end{cases}$$

Then  $f^*(\mathcal{P}_j) \geq f(Q_j)$ . Now let  $Q = Q_0 \cup Q_1 \cup Q_2 \cup \dots \cup Q_v$ . Then  $Q$  is a  $\delta$ -fine partition of  $[a, b]_{\mathbb{T}}$  and

$$(H_{\Delta}) \int_a^b f \Delta t \geq f^*(\mathcal{P}) \geq f(Q) - \sum_{j=1}^v \frac{\epsilon}{2^{k_j}} > (H_{\Delta}) \int_a^b f \Delta t - 2\epsilon.$$

Similarly, in the case  $f^*(\mathcal{P}) > (H_{\Delta}) \int_a^b f \Delta t$  we can construct a  $\delta$ -fine partition  $Q'$  of  $[a, b]_{\mathbb{T}}$  such that

$$f^*(\mathcal{P}) \leq f(Q') + \epsilon < (H_{\Delta}) \int_a^b f \Delta t + 2\epsilon.$$

Hence,  $f^*$  is  $H$ -integrable and

$$(H) \int_a^b f^* = (H_{\Delta}) \int_a^b f \Delta t.$$

□

From Theorem 2.5 and 2.6, we get the following theorem.

**THEOREM 2.7.** *A function  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  is  $H_{\Delta}$ -integrable if and only if  $f^* : [a, b] \rightarrow \mathbb{R}$  is Henstock integrable. In this case,*

$$(H_{\Delta}) \int_a^b f \Delta t = (H) \int_a^b f^*.$$

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Department of Mathematics  
Chungnam National University  
Daejeon 305-764, Republic of Korea  
*E-mail*: parkjm@cnu.ac.kr

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Department of Mathematics Education  
KongJu National University  
Kongju 314-701, Republic of Korea  
*E-mail*: dhlee@kongju.ac.kr

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Department of Mathematics Education  
Chungbuk National University  
Chungju 360-763, Republic of Korea  
*E-mail*: yoonjh@cbnu.ac.kr

\*\*\*\*

Department of Mathematics Education  
Seowon University  
Chungju 361-742, Republic of Korea  
*E-mail*: ykkim@dragon.seowon.ac.kr

\*\*\*\*\*

Department of Mathematics  
Chungnam University  
Daejeon 305-764, Republic of Korea  
*E-mail*: ssongtae@edurang.dje.go.kr